

# The Exact Solution of the Cauchy Problem for a generalized "linear" vectorial Fokker-Planck Equation - Algebraic Approach

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## Abstract

The exact solution of the Cauchy problem for a generalized "linear" vectorial Fokker-Planck equation is found using the disentangling techniques of R. Feynman and algebraic (operational) methods. This approach may be considered as a generalization of the Masuo Suzuki's method for solving the 1-dimensional linear Fokker-Planck equation.

## 1 Introduction

The Fokker-Planck equations (FPE), the one-dimensional FPE

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x} [a(t, x)W] + \frac{\partial^2}{\partial x^2} [D(t, x)W(t, x)], \quad t \geq 0, \quad x \in \mathbb{R}, \quad (1)$$

and the "vectorial" FPE

$$\frac{\partial w}{\partial t} = -\nabla \cdot [\mathbf{a}(t, \mathbf{x})w] + \nabla \nabla : \left\{ \hat{D}(t, \mathbf{x})w(t, \mathbf{x}) \right\}, \quad t \geq 0, \quad \mathbf{x} \in \mathbb{R}^n, \quad (2)$$

where  $\mathbf{a}(t, \mathbf{x}) = (a_1(t, \mathbf{x}), a_2(t, \mathbf{x}), \dots, a_n(t, \mathbf{x}))^T$  is the "drift vector",  $\hat{D}(t, \mathbf{x})$  is a symmetric non-negative definite "diffusion" tensor field of II rank, and  $\nabla \nabla : \hat{D} = \frac{\partial^2 D_{ij}}{\partial x_i \partial x_j}$  (Einstein summation convention accepted), are widely used [1]–[19] as a tool in modelling various processes in many areas of the theoretical and mathematical physics, chemistry and biology as well as in the pure and applied mathematics and in engineering: *the nonequilibrium statistical mechanics* (in particular in the theory of Brownian motion and similar phenomena: random walks, the fluctuations of the liquid surfaces, the local density fluctuations in fluids and solids, the fluctuations of currents, etc); *the metrology* (Josephson voltage standards); *the laser physics*; *the turbulence theory*;

*the cellular behaviour; the neurophysiology; the population genetics; the mathematical theory and applications of the stochastic processes,— to mention only a few of them.*

Because of its importance there have been many attempts to solve FPE exactly or approximately ( for a review see [4, 6 - 11, 14, 19]). Among the recent investigations on this problem noteworthy for us is the method of M. Suzuki [18].

In this paper we find the exact solution of following Cauchy problem:

$$\frac{\partial u}{\partial t} = a_1(t)u(t, \mathbf{x}) + \mathbf{a}_2(t) \cdot \nabla u + a_3(t)\mathbf{x} \cdot \nabla u + \hat{a}_4(t) : \nabla \nabla u, \quad u(0, \mathbf{x}) = \phi(\mathbf{x}), \quad (3)$$

where  $\hat{a}_4(t)$  is a symmetric non-negative definite tensor function of second rank of the scalar parameter  $t$ .

It is easy to see that the Eq.(3) is connected with the "linear" vectorial FPE (2) with a linear in  $\mathbf{x}$  "drift vector"  $\mathbf{a}(t, \mathbf{x}) = \mathbf{b}_1 + b_2\mathbf{x}$  and an independent of  $\mathbf{x}$  diffusion tensor  $\hat{D}$ . ( Here  $\mathbf{b}_1$ ,  $b_2$  and  $\hat{D}$  are functions of  $t$ .) Therefore the Eq. (3) is a slight generalization of the "linear" vectorial FPE (2) with  $t$ -dependent coefficients.

In the paper [20] the "isotropic" problems

$$\frac{\partial u}{\partial t} = a_1 u(t, \mathbf{x}) + \mathbf{a}_2 \cdot \nabla u + a_3 \mathbf{x} \cdot \nabla u + a_4 \Delta u, \quad u(0, \mathbf{x}) = \phi(\mathbf{x}) \quad (4)$$

and

$$\frac{\partial u}{\partial t} = a_1(t)u(t, \mathbf{x}) + \mathbf{a}_2(t) \cdot \nabla u + a_3(t)\mathbf{x} \cdot \nabla u + a_4(t)\Delta u, \quad u(0, \mathbf{x}) = \phi(\mathbf{x}) \quad (5)$$

have been exactly solved ( here  $a_4$  and  $a_4(t)$  are arbitrary non-negative constant and function of  $t$  respectively).

In the paper [21] we have found the exact solutions of the following Cauchy problems:

$$\frac{\partial u}{\partial t} = a_1 u(t, \mathbf{x}) + \mathbf{a}_2 \cdot \nabla u + a_3 \mathbf{x} \cdot \nabla u + \hat{a}_4 : \nabla \nabla u, \quad u(0, \mathbf{x}) = \phi(\mathbf{x}) \quad (6)$$

and

$$\frac{\partial u}{\partial t} = a_1(t)u(t, \mathbf{x}) + \mathbf{a}_2(t) \cdot \nabla u + a_3(t)\mathbf{x} \cdot \nabla u + a_4(t)\hat{a} : \nabla \nabla u, \quad u(0, \mathbf{x}) = \phi(\mathbf{x}) \quad (7)$$

where  $\hat{a}_4$  and  $\hat{a}$  are symmetric non-negative definite tensors of second rank and  $a_4(t)$  is a scalar function;  $a_4(t) > 0$ . ( It is obvious that the problem (3) is more general than the problem (7) : in (3),  $\hat{a}_4(t)$  is arbitrary symmetric non-negative definite tensor function of second rank, while in (7)  $\hat{a}_4(t)$  has a special form:  $\hat{a}_4(t) = a_4(t)\hat{a}$ . )

Our method may be regarded as a combination of the disentangling techniques of R. Feynman [22] with the operational methods developed in the functional analysis and in particular in the theory of pseudodifferential equations with partial derivatives [23]–[27]. As we have emphasized in [20] and [21] this approach is an extension and generalization of the M. Suzuki's method [18] for solving the one-dimensional linear FPE (1).

## 2 Exact Solution of the Cauchy Problem (3)

In view of the  $t$ -dependence of the coefficients in the Eq. (3), formally we have for the solution of the initial value problem (3) an ordered exponential

$$u(t, \mathbf{x}) = \left( \exp_+ \int_0^t [a_1(s) + \mathbf{a}_2(s) \cdot \nabla + a_3(s) \mathbf{x} \cdot \nabla + \hat{a}_4(s) : \nabla \nabla] ds \right) \phi(\mathbf{x}), \quad (8)$$

where

$$\begin{aligned} \exp_+ \int_0^t \hat{C}(s) ds &\equiv T\text{-}\exp \int_0^t \hat{C}(s) ds \\ &= \hat{1} + \lim_{k \rightarrow \infty} \sum_{n=1}^k \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \hat{C}(t_1) \hat{C}(t_2) \dots \hat{C}(t_n). \end{aligned} \quad (9)$$

If we introduce the operators

$$\hat{A}(t) = \mathbf{a}_2(t) \cdot \nabla + a_3(t) \mathbf{x} \cdot \nabla \quad \text{and} \quad \hat{B}(t) = \hat{a}_4(t) : \nabla \nabla, \quad (10)$$

we may write (8) in the form

$$u(t, \mathbf{x}) = e^{\int_0^t a_1(s) ds} \left( \exp_+ \int_0^t [\hat{A}(s) + \hat{B}(s)] ds \right) \phi(\mathbf{x}), \quad (11)$$

as the first term in the exponent commutes with all others.

To proceed with the pseudodifferential operator in Eq. (11) we shall use the theorem of M.Suzuki [18] :

If

$$[\hat{A}(t), \hat{B}(s)] = \alpha(t, s) \hat{B}(s),$$

then

$$\exp_+ \int_0^t [\hat{A}(s) + \hat{B}(s)] ds = \left( \exp_+ \int_0^t \hat{A}(s) ds \right) \left( \exp_+ \int_0^t \hat{B}(s) e^{-\int_0^s \alpha(u, s) du} ds \right). \quad (12)$$

In our case we have

$$\begin{aligned} [\hat{A}(s), \hat{B}(s')] &\equiv [\mathbf{a}_2(s) \cdot \nabla + a_3(s) \mathbf{x} \cdot \nabla, \hat{a}_4(s') : \nabla \nabla] \\ &= -2a_3(s) \hat{a}_4(s') : \nabla \nabla \equiv -2a_3(s) \hat{B}(s'). \end{aligned} \quad (13)$$

Therefore from (12) we obtain

$$\exp_+ \int_0^t [\hat{A}(s) + \hat{B}(s)] ds = \left( \exp_+ \int_0^t \hat{A}(s) ds \right) \left( \exp_+ \int_0^t \hat{B}(s) e^{2 \int_0^s a_3(u) du} ds \right). \quad (14)$$

The linearity of the integral and the explicit form of  $\hat{A}$  (see Eq. (10)) permit to write the first factor in (14) in terms of usual, not ordered, operator valued exponent

$$\exp_+ \int_0^t \hat{A}(s) ds \equiv \exp_+ \int_0^t [\mathbf{a}_2(s) \cdot \nabla + a_3(s) \mathbf{x} \cdot \nabla] ds = e^{\vec{a}_2(t) \cdot \nabla + a_3(t) \mathbf{x} \cdot \nabla}. \quad (15)$$

For convenience we introduce the following notations:

$$\alpha_1(t) = \int_0^t a_1(s)ds, \quad \vec{\alpha}_2(t) = \int_0^t \mathbf{a}_2(s)ds, \quad \alpha_3(t) = \int_0^t a_3(s)ds. \quad (16)$$

Consequently (from now on "'' means  $\frac{d}{dt}$ )

$$\begin{aligned} \alpha'_1(t) &= a_1(t), \quad \vec{\alpha}'_2(t) = \mathbf{a}_2(t), \quad \alpha'_3(t) = a_3(t), \\ \alpha_1(0) &= 0, \quad \vec{\alpha}_2(0) = \mathbf{0}, \quad \alpha_3(0) = 0. \end{aligned} \quad (17)$$

Thus we obtain from the Eq. (11)

$$u(t, \mathbf{x}) = e^{\alpha_1(t)} e^{[\vec{\alpha}_2(t) + \alpha_3(t)\mathbf{x}] \cdot \nabla} \left( \exp_+ \int_0^t \hat{a}_4(s) e^{2\alpha_3(s)} : \nabla \nabla ds \right) \phi(\mathbf{x}). \quad (18)$$

Finally using the formulae (see [20] and [21])

$$\begin{aligned} & \left[ \exp_+ \left( \int_0^t \hat{\Psi}(s) : \nabla \nabla ds \right) \right] \phi(\mathbf{x}) \\ &= \frac{1}{\sqrt{\det(4\pi\hat{\tau}(t))_{\mathbb{R}^n}}} \int \left\{ \exp \left[ -(\mathbf{x} - \mathbf{y}) \cdot \frac{\hat{\tau}^{-1}(t)}{4} \cdot (\mathbf{x} - \mathbf{y}) \right] \right\} \phi(\mathbf{y}) d\mathbf{y}, \end{aligned} \quad (19)$$

where

$$d\mathbf{y} = dy_1 dy_2 \dots dy_n, \quad \hat{\tau}(t) = \int_0^t \hat{\Psi}(s) ds$$

and

$$e^{\vec{\alpha}_2(t) \cdot \nabla + \alpha_3(t)\mathbf{x} \cdot \nabla} g(\mathbf{x}) = g \left( \mathbf{x} e^{\alpha_3(t)} + \int_0^t \mathbf{a}_2(s) e^{\alpha_3(s)} ds \right) \equiv g(\mathbf{z}), \quad (20)$$

we find from the Eq. (18) the following expression for the exact solution of the Cauchy problem (3) ( $\hat{\Psi}(s) = \hat{a}_4(s) \exp[2a_3(s)]$ ) :

$$u(t, \mathbf{x}) = \frac{e^{\alpha_1(t)}}{\sqrt{\det(4\pi\hat{\tau}(t))_{\mathbb{R}^n}}} \int \left\{ \exp \left[ -(\mathbf{z} - \mathbf{y}) \cdot \frac{\hat{\tau}^{-1}(t)}{4} \cdot (\mathbf{z} - \mathbf{y}) \right] \right\} \phi(\mathbf{y}) d\mathbf{y}, \quad (21)$$

where

$$\hat{\tau}(t) = \int_0^t \hat{a}_4(s) e^{2\alpha_3(s)} ds$$

is a symmetric non-negative definite second rank tensor function of  $t$ ,  $d\mathbf{y} = dy_1 \dots dy_n$  and  $\mathbf{z}$  is defined in (20).

Substituting the expression (21) in the Eq. (3) we see immediately that the function  $u(t, \mathbf{x})$  is a solution of the problem (3), and, according to the Cauchy theorem, it is the only classical solution of this problem.

### 3 Concluding remarks

- The exact solutions of the Cauchy problem (3) is obtained using the algebraic method we have described.
- When  $\hat{a}_4(t)$  is scalar:  $\hat{a}_4(t) = a_4(t)\hat{1}$  ( in this case  $\hat{a}_4 : \nabla\nabla = a_4\Delta$  ) the "anisotropic" problem (3) turns to the "isotropic" one, with the exact solution found in [20]. It is easy to check that the solution (21) turns to the solution obtained in [20] (there is an error in [20]: the sign before  $\mathbf{a}_2$  in the Eqs. (17) and (34) there, should be (+)).
- In the case  $\hat{a}_4(t) = a_4(t)\hat{a}$  the Cauchy problem (3) reduces to the problem (7) treated in [21]. In this case the solution (21) turns to the solution obtained in [21].
- For different choices of the coefficients  $a_j$  and  $\mathbf{a}_2$  the Eq. (3) may be regarded also as a set of different diffusion equations. Therefore from the formula (21) we obtain the exact solutions of the Cauchy problems for this set of diffusion equations.

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